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RIEMANNIAN SUBMERSIONS COMMUTING WITH THE LAPLACIAN

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1. Introduction

Let M and N be smooth Riemannian manifolds. Let $\Delta_M^p = d\delta + \delta d: \bigwedge^p(M) \to \bigwedge^p(M)$ denote the Laplace-Beltrami operator on the differential p-forms of M. Define the set

$$\Omega^p(M, N) = \{\varphi: M \to N | \varphi \text{ is a smooth surjective mapping with rank } \varphi_* \geq 1 \text{ and } \varphi^* \Delta_N^p A = \Delta_M^p \varphi^* A \text{ for all } A \in \bigwedge^p(N)\}$$

of pth Laplacian-commuting mappings. If $\Omega^p(M, N)$ is empty, it is said to be trivial. The condition on the rank is not necessary in defining $\Omega^0(M, N)$ because any surjective mapping $\varphi: M \to N$ with $\varphi^* \Delta_N f = \Delta_M \varphi^* f$ for all smooth functions f on N satisfies rank $\varphi_* = n = \dim N$. In this paper, we ask for the mappings contained in $\Omega^p(M, N)$. Watson [4] showed that $\varphi: M \to N$ is contained in $\Omega^0(M, N)$ if and only if it is a harmonic Riemannian submersion. He also proved that the nontriviality of $\Omega^p(M, N)$, $p \geq 0$, implies that the elements of $\Omega^p(M, N)$ are Riemannian submersions. We therefore ask for the Riemannian submersions which commute with the Laplacian. It is an immediate consequence of our main result that $\Omega^1(M, N) = \Omega^2(M, N) = \cdots = \Omega^n(M, N)$.

In § 2, the basic facts of a Riemannian submersion will be described, especially its structure tensor. Several relations between the curvature tensors of M and N and the structure tensor are given in § 3 .The set $\Omega^1(M, N)$ is studied in § 4, and in the last section the set $\Omega^p(M, N)$, $p \geq 2$, is examined.

2. Riemannian submersions

Let M (resp. N) be an m (resp. n)-dimensional manifold with Riemannian metric ds_M^2 (resp. ds_N^2), and let $\varphi: M \to N$ be a Riemannian submersion. Then we may assume $n < m$; for, if $m = n$, a Riemannian submersion (Riemannian covering) commutes with the Laplacian [4]. We choose local forms $\omega_1, \cdots, \omega_m$ on M and $\theta_1, \cdots, \theta_n$ on N such that $ds_M^2 = \Sigma \omega_a^2$, $ds_N^2 = \Sigma \theta_i^2$, and

$$\varphi^*(\theta_i) = \omega_i\,, \qquad i = 1, \cdots, n\,. \tag{2.1}$$

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(In the sequel, the indices $i, j, k, \cdots$ run from 1 to n; $a, b, c, \cdots$ from 1 to m, and $\alpha, \beta, \gamma, \cdots$ from $n+1$ to m.)

The structure equations of M are

$$(2.2) \qquad d\omega_a = \Sigma\omega_b \wedge \omega_{ba} , \qquad d\omega_{ab} = \Sigma\omega_{ac} \wedge \omega_{cb} - \tfrac{1}{2}\Sigma R_{abcd}\omega_c \wedge \omega_d ,$$

where $\omega_{ab} = -\omega_{ba}$ and the R_{abcd} are the components of its curvature tensor. The components of the curvature tensor of N will be denoted by K_{ijkl}.

Taking the exterior derivative of (2.1), we get

$$\Sigma\omega_j \wedge (\varphi^*\theta_{ji} - \omega_{ji}) - \Sigma\omega_\alpha \wedge \omega_{\alpha i} = 0 .$$

This allows us to put

$$(2.3) \qquad \omega_{ji} - \varphi^*\theta_{ji} = \Sigma L_{jia}\omega_a , \qquad \omega_{i\alpha} = \Sigma L_{i\alpha a}\omega_a ,$$

where $L_{ijk} = 0$, $L_{ij\alpha} = -L_{ji\alpha}$, $L_{ij\alpha} = L_{i\alpha j}$ and $L_{i\alpha\beta} = L_{i\beta\alpha}$. In the sequel, we will drop φ^* from such formulas when its presence is clear from the context. We call the tensor, whose components are the L_{iab}, the *structure tensor* of φ. If $\Sigma L_{iaa} = 0$, that is, if $\Sigma L_{i\alpha\alpha} = 0$ (resp. $L_{i\alpha\beta} = 0$), φ is called a *harmonic* (resp. *totally geodesic*) mapping.

The inverse image $\varphi^{-1}(x)$ of a point x of N is said to be a *fibre* of φ. A fibre of φ is a closed submanifold of M of dimension $m - n$. It is evident that $\omega_1 = \cdots = \omega_n = 0$ on the fibres, and that the restriction of $\Sigma\omega_\alpha^2$ to a fibre gives the induced Riemannian metric. The $L_{i\alpha\beta}$ may be regarded as the *second fundamental forms* of the submanifold $\varphi^{-1}(x)$. Hence, if $\Sigma L_{i\alpha\alpha} = 0$ (resp. $L_{i\alpha\beta} = 0$), then $\varphi^{-1}(x)$ is a minimal (resp. totally geodesic) submanifold of M. Suppose M is complete. Then M becomes a fibre space in Ehressman's sense. If, moreover, the fibres are totally geodesic, φ: $M \to N$ is a fibre bundle with structural group the Lie group of isometries of a fibre [1], [2]. The horizontal distribution, which is defined by $\omega_{n+1} = \cdots = \omega_m = 0$, is integrable if the $L_{ij\alpha} = 0$. If M is complete, and the $L_{i\alpha\beta}$ and $L_{ij\alpha}$ vanish, then M is locally the Riemannian product of a fibre $\varphi^{-1}(x)$ (x is any fixed point of N) and N, that is, there is an open covering $\{U_A\}$ of N such that $\varphi^{-1}(U_A)$ is isometric to the Riemannian product $\varphi^{-1}(x) \times U_A$.

3. The covariant differential of the structure tensor

The components L_{iabc} of the covariant differential of the structure tensor L_{iab} are given by

$$(3.1) \qquad \Sigma L_{iabc}\omega_c = dL_{iab} + \Sigma L_{jab}\theta_{ji} + L_{icb}\omega_{ca} + \Sigma L_{iac}\omega_{cb} .$$

This yields, in particular, by means of (2.3),

$$L_{ijka} = -\Sigma(L_{ik\alpha}L_{ja\alpha} + L_{ij\alpha}L_{ka\alpha}) . \tag{3.2}$$

Differentiating (2.3) and using the structure equations (2.2), as well as their analogues in N, we get

$$L_{iabc} - L_{iacb} = R_{iabc} - \Sigma\delta_{aj}\delta_{bl}\delta_{ck}K_{ijlk} . \tag{3.3}$$

From this and (3.2) it follows that

$$R_{ijkl} - K_{ijkl} = \Sigma(L_{il\alpha}L_{jk\alpha} - L_{ik\alpha}L_{jl\alpha} + 2L_{ij\alpha}L_{lk\alpha}) . \tag{3.4}$$

Contracting (3.3), we obtain

$$\Sigma(L_{ijaa} - L_{iaaj}) = R_{ij} - K_{ij} , \qquad \Sigma(L_{i\alpha aa} - L_{iaa\alpha}) = R_{i\alpha} ,$$

where R_{ab} (resp. K_{ij}) is the Ricci tensor given by ΣR_{acbc} (resp. ΣK_{ikjk}). Since $\Sigma L_{iaa} = 0$ implies $\Sigma L_{iaab} = 0$, the above equations lead us to

Lemma 1. *If φ is a harmonic mapping, then*

$$\Sigma L_{ijaa} = R_{ij} - K_{ij} , \qquad \Sigma L_{i\alpha aa} = R_{i\alpha} . \tag{3.5}$$

If the $L_{ij\alpha}$ vanish, then the L_{iabc} have a simple form. In fact, from (3.1) we get

Lemma 2. *If the $L_{ij\alpha} = 0$, then,*

$$L_{ijka} = 0 , \quad L_{ij\alpha k} = 0 , \quad L_{ij\alpha\beta} = -\Sigma_{i\alpha\gamma}L_{j\beta\gamma} . \tag{3.6}$$

4. The Laplacian on functions and 1-forms

In this section we study the set $\Omega^1(M, N)$. The sets $\Omega^p(M, N)$, $p \geq 2$, will be discussed in the next section.

The following lemma is useful in finding conditions for a mapping to commute with the Laplacian.

Lemma 3. *Let x be a point of N. For given $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq k \leq n$, there exists a smooth p-form $A = \Sigma A_{j_1\cdots j_p}\theta_{j_1} \wedge \cdots \wedge \theta_{j_p}$, where the sum is taken over all $j_1, \cdots, j_p$ with $j_1 < \cdots < j_p$, such that $A_{j_1\cdots j_p}(x) = 0$, $A_{i_1\cdots i_p,k}(x) = 1$ and all other $A_{j_1,\cdots,j_p,l}$ vanish. The $A_{j_1,\cdots,j_p,l}$ are the coefficients of the covariant differential of A.*

Proof. Let $(\{x_i\}, U)$ be a normal coordinate system at x, and let V be an open subset of U. For given constants $C_0, C_1, \cdots, C_n$, there is a smooth function h on N satisfying $h(x) = C_0$, $\partial h/\partial x_i(x) = C_i$, $i = 1, \cdots, n$, and $h = 0$ on $M - V$. Since $\{x_i\}$ is a normal coordinate system, covariant differentiation at x with respect to $\partial/\partial x^i$ is identical with ordinary partial differentiation. Thus a smooth p-form can be constructed whose covariant differential takes arbitrarily given values at x. The desired result now follows easily.

Let f be a smooth function on N, and put $df = \Sigma f_i\theta_i$. The covariant differential of df is given by $\Sigma f_{ij}\theta_j = df_i + \Sigma f_j\theta_{ji}$. Then $\Delta_N f = -\Sigma f_{ii}$. Similarly,

$\Delta_M\varphi^*f = -\Sigma f_{ii} - \Sigma f_j L_{j\alpha\alpha}$. The commutation condition $\varphi^*\Delta_N f = \Delta_M\varphi^* f$ may then be expressed by $\Sigma f_j L_{j\alpha\alpha} = 0$. Applying Lemma 3, we obtain

Theorem 1. *Let φ be a smooth mapping from M onto N. For any smooth function f on N, $\Delta_M\varphi^*f = \varphi^*\Delta_N f$ if and only if φ is a harmonic Riemannian submersion.*

This was first proved by Watson [4].

Let $A = \Sigma A_i\varphi_i$ be a 1-form on N. The components A_{ij} of the covariant differential $\nabla_N A$ are given by $\Sigma A_{ij}\theta_j = dA_i + \Sigma A_j\theta_{ji}$, and the components A_{ijk} of the second covariant differential $\nabla_N^2 A$ of A are given by $\Sigma A_{ijk}\theta_k = dA_{ij} + \Sigma A_{kj}\theta_{ki} + \Sigma A_{ik}\theta_{kj}$. Set $\varphi^*A = \Sigma \tilde{A}_a\omega_a$ and $\nabla_M\varphi^*A = \Sigma\tilde{A}_{ab}\omega_a \wedge \omega_b$. Then $\tilde{A}_i = A_i$, $\tilde{A}_\alpha = 0$, $\tilde{A}_{ij} = A_{ij}$, $\tilde{A}_{i\alpha} = \Sigma A_j L_{ji\alpha}$, $\tilde{A}_{\alpha i} = \Sigma A_j L_{j\alpha i}$, $\tilde{A}_{\alpha\beta} = \Sigma A_j L_{j\alpha\beta}$, $i = 1, \cdots, n; \alpha = n+1, \cdots, m$. Moreover, the components of $\nabla_M^2\varphi^*A$ are

$$
\begin{aligned}
\tilde{A}_{ijk} &= A_{ijk} + \Sigma A_l L_{lijk}\ , \\
\tilde{A}_{i\alpha\beta} &= \Sigma A_l L_{li\alpha\beta} + \Sigma A_{il} L_{l\alpha\beta}\ , \\
\tilde{A}_{\alpha ij} &= \Sigma A_l L_{l\alpha ij} + \Sigma A_{lj} L_{l\alpha i} + \Sigma A_{li} L_{l\alpha j}\ , \\
\tilde{A}_{\alpha\beta\gamma} &= \Sigma A_l L_{l\alpha\beta\gamma}\ .
\end{aligned}
\tag{4.1}
$$

To deduce the first equation of (4.1), we use (3.2). Since $\Delta_M\varphi^*A = -\Sigma(\tilde{A}_{abb} - \tilde{A}_b R_{ba})\omega_a$and $\varphi^*\Delta_N A = -\Sigma(A_{ijj} - A_j K_{ji})\omega_i$, formula (4.1) yields

Lemma 4.

$$
\begin{aligned}
\Delta_M\varphi^*A - \varphi^*\Delta_N A = {} & \Sigma\{A_j(R_{ji} - K_{ji} - \Sigma L_{ji a a}) - A_{ij}\Sigma L_{j\alpha\alpha}\}\omega_i \\
& + \Sigma\{A_j(R_{j\alpha} - \Sigma L_{j\alpha a a}) - 2\Sigma A_{ji} L_{ji\alpha}\}\omega_\alpha\ .
\end{aligned}
\tag{4.2}
$$

We introduce the operator $H\colon \wedge^1(M) \to \wedge^1(M)$ defined by $H(\Sigma B_a\omega_a) = \Sigma B_i\omega_i$. This definition does not depend on the choice of the local forms ω_α. Using Lemmas 1 and 3, we obtain from Lemma 4

Proposition 1. *Let $\varphi\colon M \to N$ be a Riemannian submersion. For any 1-form A on N, $H(\Delta_M\varphi^*A) = \varphi^*\Delta_N A$ if and only if φ is a harmonic Riemannian submersion.*

If $\Delta_M\varphi^*A = \varphi^*\Delta_N A$ for any 1-form A, then φ is harmonic, and $\Sigma L_{j\alpha a a} = R_{j\alpha}$ by Lemma 1. Hence the coefficient of ω_α in (4.2) vanishes if and only if the $L_{ji\alpha} = 0$. Conversely, if $\Sigma L_{i\alpha\alpha} = 0$ and the $L_{ij\alpha} = 0$, then (4.2) implies $\Delta_M\varphi^*A = \varphi^*\Delta_N A$ for any 1-form A. Thus we have

Proposition 2. *Let $\varphi\colon M \to N$ be a smooth surjective mapping with rank $\varphi_* \geq 1$. For any 1-form A on N, $\Delta_M\varphi^*A = \varphi^*\Delta_N A$ if and only if φ is a harmonic Riemannian submersion and the $L_{ij\alpha}$ vanish.*

5. The Laplacian on p-forms

Let $A = \Sigma A_{i_1\cdots i_p}\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$ be a p-form on N, and set $\varphi^*A = \Sigma\tilde{A}_{a_1\cdots a_p}\omega_{a_1} \wedge \cdots \wedge \omega_{a_p}$. Then $\tilde{A}_{i_1\cdots i_p} = A_{i_1\cdots i_p}$, and all other components vanish. Denote the components of $\nabla_N A$ (resp. $\nabla_M\varphi^*A$) by $A_{i_1\cdots i_p,j}$ (resp. $\tilde{A}_{a_1\cdots a_p,b}$) and the components of $\nabla_N^2 A$ (resp. $\nabla_M^2\varphi^*A$) by $A_{i_1\cdots i_p,jk}$ (resp. $\tilde{A}_{a_1\cdots a_p,bc}$). We have

$$\Delta_N A = -\sum\Big(\sum_j A_{i_1\cdots i_p,jj} - \sum_{\rho=1}^{p}\sum_j A_{i_1\cdots i_{\rho-1}ji_{\rho+1}\cdots i_p}K_{ji_\rho} + \sum_{\rho\neq\sigma}^{p}\sum_{i,j} A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_{\sigma-1}ji_{\sigma+1}\cdots i_p}K_{ii_\rho ji_\sigma}\Big)\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}\,.$$

as well as a similar expression for $\Delta_M\varphi^*A$. Put

$$\Delta_M\varphi^*A - \varphi^*\Delta_N A = \Sigma B_{a_1\cdots a_p}\omega_{a_1}\wedge\cdots\wedge\omega_{a_p}\,. \tag{5.1}$$

As in the previous section, $\tilde{A}_{a_1\cdots a_p,bc}$ can be expressed in terms of the $A_{i_1\cdots i_p}$, $A_{i_1\cdots i_p,j}$, $A_{i_1\cdots ip,jk}$, L_{iab} and L_{iabc}. For example,

$$\tilde{A}_{i_1\cdots i_p,ij} = A_{i_1\cdots i_p,ij} - \sum_{\rho=1}^{p}\sum_{j,\alpha} A_{i_1\cdots i_{\rho-1}ki_{\rho+1}\cdots i_p}(L_{ij\alpha}L_{ki_\rho\alpha} - L_{ki\alpha}L_{i_\rho j\alpha})\,.$$

Employing relations of this type, we get

Lemma 5. *The coefficients in* (5.1) *may be expressed as*

$$\begin{aligned} B_{i_1\cdots i_p} &= \sum_{\rho=1}^{p}\sum_i A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_p}\Big(R_{ii_\rho} - K_{ii_\rho} - \sum_a L_{ii_\rho aa}\Big) \\ &\quad + \sum_{\rho\neq\sigma}^{p}\sum_{\substack{i,j\\ \alpha}} A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_{\sigma-1}ji_{\sigma+1}\cdots i_p}L_{ij\alpha}L_{i_\rho i_\sigma\alpha} - \sum_{i,\alpha} A_{i_1\cdots i_p,i}L_{i\alpha\alpha}\,, \end{aligned} \tag{5.2}$$

$$\begin{aligned} B_{i_1\cdots i_{\rho-1}\alpha i_{\rho+1}\cdots i_p} &= \sum_i A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_p}\Big(R_{i\alpha} - \sum_a L_{i\alpha aa}\Big) \\ &\quad - 2\sum_{\sigma=1}^{p}\sum_j A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_{\sigma-1}ji_{\sigma+1}\cdots i_p}\Big(R_{i\alpha ji_\sigma} + \sum_\beta L_{i\alpha\beta}L_{ji_\sigma\beta}\Big) \\ &\quad - 2\sum_{i,j} A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_p,j}L_{ij\alpha}\,, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &B_{i_1\cdots i_{\rho-1}\alpha i_{\rho+1}\cdots i_{\sigma-1}\beta i_{\sigma+1}\cdots i_p} \\ &\quad = -\sum_{i,j} A_{i_1\cdots i_{\rho-1}ii_{\rho+1}\cdots i_{\sigma-1}ji_{\sigma+1}\cdots i_p}\Big(R_{ij\alpha\beta} + 2\sum_a L_{i\alpha a}L_{j\beta a}\Big)\,, \end{aligned} \tag{5.4}$$

$$B_{a_1\cdots\alpha\cdots\beta\cdots a_p} = 0\,. \tag{5,5}$$

If for any p-form A, the corresponding $B_{i_1\cdots i_p}$ vanish, then from (5.2) and Lemma 3 we have $\Sigma L_{i\alpha\alpha} = 0$. If, in addition, the $B_{i_1\cdots i_{\rho-1}\alpha i_{\rho+1}\cdots i_p} = 0$, then (5.3) implies that the $L_{ij\alpha} = 0$. Conversely, assume $\Sigma L_{i\alpha\alpha} = 0$ and the $L_{ij\alpha} = 0$. Then by Lemmas 1 and 2 we conclude that the $B_{a_1\cdots a_p} = 0$ for any p-form A. Taking account of Proposition 2, we obtain

Theorem 2. *Let $\varphi\colon M\to N$ be a smooth surjective mapping with rank $\varphi_*\geq 1$. Let $p\,(\geq 1)$ be fixed. For any p-form A, $\Delta_M\varphi^*A = \varphi^*\Delta_N A$ if and only if $\varphi\colon M\to N$ is a harmonic Riemannian submersion with integrable horizontal distribution.*

Corollary 1. $\Omega^1(M, N) = \Omega^2(M, N) = \cdots = \Omega^n(M, N)$.

It was shown in [4] that if $\Omega^p(M, N)$ is nontrivial for a fixed p, then $b_p(N) \leq b_p(M)$, where b_p denotes the p-th betti number. Thus

Corollary 2. *Let ϕ: $M \to N$ be a smooth surjective mapping with rank $\phi_* \geq 1$. Then a necessary condition that ϕ be a harmonic Riemannian submersion with integrable horizontal distribution is $b_p(N) \leq b_p(M)$ for all $p = 1, \cdots, n$.*

Bibliography

[1] R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle,* Proc. Amer. Math. Soc. **11** (1960) 236–242.

[2] T. Nagano, *On fibred Riemann manifolds,* Sci. Rep. Coll. Gen. Ed. Univ. Tokyo **10** (1960) 17–27.

[3] B. O'Neill, *The fundamental equations of a submersion,* Mich. Math. J. **13** (1966) 459–469.

[4] B. Watson, *Manifold maps commuting with the Laplacian,* J. Differential Geometry **8** (1973) 85–94.

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